Unclassified Ordinary Differential Equations

1. $yy' + x^2 = 0$
2. $y' + x^2 y = 0$
3. $dy = (y + \sin(x))dx$

Solutions

1. Ordinary differential equation with separable variables:

$$\frac{dy}{dx}y = -x^2$$

$$dyy = -x^2 dx$$

Integrate both sides:

$$\frac{y^2}{2} = -\frac{x^3}{3} + C$$

Therefore we have:

$$y = \sqrt{-2\frac{x^3}{3} + C}$$

$$y = -\sqrt{-2\frac{x^3}{3} + C}$$

2. Ordinary differential equation with separable variables:

$$\frac{dy}{dx} = -x^2 y$$

$$\frac{dy}{y} = -x^2 dx$$

Integrate both sides

$$\ln(y) = \frac{-x^3}{3} + C$$

Therefore:

$$y = e^{\frac{-x^3}{3} + C}$$

3. Ordinary linear differential equation ($y' + P(x)y = Q(X)$)
Rewrite:

$$y' - \sin(x) = y$$

We propose the substitution: $y = uv$, so $y' = u'v + uv'$:

$$u'v + uv' - \sin(x) = uv$$

$$u'v + uv' - uv = \sin(x)$$

$$v(u' - u) + uv' = \sin(x)$$

This can be solved as a system, where we have $v(u' - u) = 0$ and $uv' = \sin(x)$. Solving the first equation:

$$u' - u = 0$$

$$\frac{du}{dx} = u$$

$$\frac{du}{u} = dx$$

Integrating both sides:

$$\ln(u) = x$$

$$u = e^x$$

We solve the other differential equation:

$$e^x v' = \sin(x)$$

$$e^x \frac{dv}{dx} = \sin(x)$$

$$dv = \sin(x)e^{-x}dx$$

Integrating the right-hand side:

$$\int \sin(x)e^{-x}dx$$

We can perform integration by parts, we then propose: $dh = e^{-x}$ and $j = \sin(x)dx$, so we have, $dj = \cos(x)$ and $h = -e^{-x}$.

$$\int \sin(x)e^{-x}dx = \sin(x)(-e^{-x}) - \int(-e^{-x})(\cos(x))dx$$

Distributing signs:

$$\int \sin(x)e^{-x}dx = -\sin(x)(e^{-x}) + \int(e^{-x})(\cos(x))dx$$

Apply integration by parts again

$$\int(\cos(x))(e^{-x})dx = \cos(x)(-e^{-x}) - \int(-\sin(x))(-e^{-x})dx$$

Therefore:

$$\int \sin(x)e^{-x}dx = -\sin(x)(e^{-x}) + \left[\cos(x)(-e^{-x}) - \int(-\sin(x))(-e^{-x})dx\right]$$

Distributing signs:

$$\int \sin(x)e^{-x}dx = -\sin(x)(e^{-x}) - \cos(x)(e^{-x}) - \int(\sin(x))(e^{-x})dx$$

Regroup:

$$2\int \sin(x)e^{-x}dx = -\sin(x)(e^{-x}) - \cos(x)(e^{-x})$$

$$\int \sin(x)e^{-x}dx = \frac{-\sin(x)(e^{-x}) - \cos(x)(e^{-x})}{2}$$

Going back to the differential equation:

$$v = \frac{-\sin(x)(e^{-x}) - \cos(x)(e^{-x})}{2} + C$$

Having values of v and u, I can find y:

$$y = \left(\frac{-\sin(x)(e^{-x}) - \cos(x)(e^{-x})}{2} + C\right)e^x$$